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Theory of weakly damped Stokes waves: a new formulation and its physical interpretation

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A tractable theory for weakly damped, nonlinear Stokes waves on deep water was recently formulated by Ruvinsky & Friedman (1985*a*, *b*; 1987). In this paper we show how the theory can be simplified, and that it is equivalent to a boundary-layer model for surface waves proposed by Longuet-Higgins (1969), when the latter is generalized to include surface tension and nonlinearity. The potential part of the flow is determined by boundary conditions applied at the base of the vortical boundary layer. The theory may be of use in discussing the generation of waves by wind.

1. Introduction

Since Stokes's original paper (1847) the irrotational theory of surface waves on water of infinite or uniform finite depth has been outstandingly successful in predicting many observed wave phenomena. For certain applications, however, viscous damping of the waves is important, and it would be highly convenient to have equations and boundary conditions of comparable simplicity as for undamped waves. A first step in this direction was made by Lamb (1932) who showed that for most wavelengths of interest the effects of viscosity on linear, deep-water waves are confined to a thin vortex layer near the free surface, of thickness $D_0 = (2\nu/\sigma)^{\frac{1}{2}}$ (where ν denotes the kinematic viscosity and σ the radian frequency). When $kD_0 \ll 1$ (k the wavenumber) we may say that the waves are weakly damped. Lamb (1932) calculated the tangential stress at the surface that would be required in a perfectly periodic state; hence the energy loss and consequent wave damping in the absence of such applied stresses.

Longuet-Higgins (1960) considered the action of a general, tangential stress at the free surface, varying sinusoidally in the horizontal direction.† He showed that the stress would tend to produce a vortical boundary layer that was thicker at points 90° out of phase with the stress. For example, a stress greatest at the wave crest would produce a thickening of the layer on the rear wave slopes, tending to pump energy into the potential flow in the interior. Similarly, in the absence of any wind the viscous stresses at the base of the vortical layer would tend to thicken the layer on the *forward* slopes of the wave and to produce the calculated wave damping (see figure 1.)

In several papers Ruvinsky & Freidman (1985*a*, *b*; 1987) have independently formulated a system of equations for weakly damped surfaces waves in deep water,

† At second order in the wave steepness it is known that vorticity may diffuse into the interior of the fluid (see Longuet-Higgins 1953, 1960). Here we confine attention mainly to the linear theory, or at least to times after the start of the motion that are short enough for the diffusion or convection of vorticity to be still negligible.

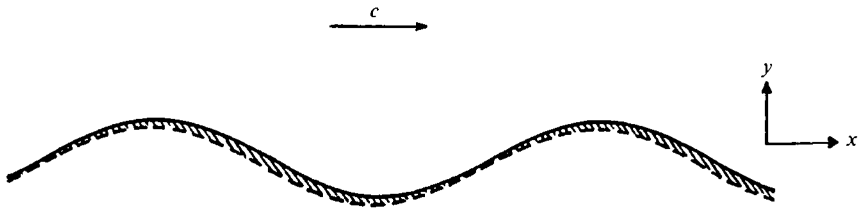

FIGURE 1. Sketch of the vortical boundary-layer induced by viscous stresses in a deep-water wave.

and have successfully applied it to the calculation of capillary–gravity ripples riding on the forward face of steep gravity waves. They formally separated the velocity field into its irrotational and vortical components and hence were led to the coupled system: (3.1) to (3.5) below. This they then solved for periodic waves by numerical integration. Their analysis is summarized in the Appendix to a recent paper (Ruvinsky, Feldstein & Freidman, 1991). As a final remark they state that it is possible to use a certain 'simpler set of equations', but they give no derivation or physical explanation. They justify the simpler system solely on the grounds that it yields the correct expression for the decay of weakly damped surface waves.

The purpose of the present note is, first, to give an analytical derivation of this simpler set of equations and, secondly, to provide a physical explanation for them. Indeed we show that the simpler equations express precisely the physical argument given by Longuet-Higgins (1969).

In a further discussion (§4 below) we point out that the simplified system of equations may be generalized so as to include applied surfaces stresses. Thus it may be of use in the theory of wave generation by wind.

2. Dynamics of the vortical layer

In this section we summarize the physical argument given by Longuet-Higgins (1969) for gravity waves and extend it to include capillarity.

Consider a surface wave travelling to the right with speed c as in figure 1. Let n and s denote coordinates normal and tangential to the free surface, and v', u' the vortical components of the orbital velocities v, u in the corresponding directions. If D denotes the thickness of the vortical layer and $M = \int \rho u' \,\mathrm{d}n$, the integrated mass flux across the layer, we have by continuity

$$\rho\frac{\partial D}{\partial t} = -\frac{\partial M}{\partial s} = \frac{1}{c}\frac{\partial M}{\partial t} = \frac{\tau}{c}, \tag{2.1}$$

where τ is the tangential stress acting on the layer. If τ is proportional to $\exp \mathrm{i}\,(ks-\sigma t)$ where k is the wavenumber, then integration of (2.1) with respect to the time t gives

$$D = \frac{\mathrm{i}\tau}{\rho\sigma c} + \text{constant}. \tag{2.2}$$

Thus D leads τ by 90°. Now in the case when the tangential stress at the surface vanishes, the only other tangential force acting on the boundary layer is the viscous force at the base of the boundary layer, which is given by

$$-\tau = 2\mu\frac{\partial v}{\partial s} = 2\mu\frac{\partial^2 \eta}{\partial s\,\partial t} = 2\mu\sigma k\eta, \tag{2.3}$$

where μ is the coefficient of viscosity and η is the surface elevation. From (2.2) and (2.3) we see that the tangential stress produces an additional surface elevation

$$\eta' = -\frac{2\mathrm{i}\mu k}{\rho c}\eta. \tag{2.4}$$

This produces an added normal stress

$$\delta_1 p = \left(\rho g - T\frac{\partial^2}{\partial s^2}\right)\eta', \tag{2.5}$$

where g and T denote gravity and surface tension. But from the dispersion relation

$$\sigma^2 = gk + (T/\rho)\, k^3 \tag{2.6}$$

for capillary–gravity waves, equation (2.5) can be written

$$\delta_1 p = \frac{\rho\sigma}{k}\eta' = -2\mathrm{i}\mu\sigma k\eta. \tag{2.7}$$

In addition we must take into account the viscous component of the normal stress p_{nn} at the surface, which may be written

$$2\mu\frac{\partial v}{\partial n} = 2\mu k v = -2\mathrm{i}\mu\sigma k\mu. \tag{2.8}$$

Adding this to (2.5) we find that it doubles the total pressure, giving altogether

$$\delta p = -4\mathrm{i}\mu\sigma k\eta. \tag{2.9}$$

Clearly δp is greatest on the forward face of the wave where the orbital velocity is upwards. Hence δp does work against the orbital motion and so damps the waves. In fact (2.9) leads to the classical law for viscous decay of waves of amplitude a, namely

$$a \propto \exp(-2\nu k^2 t). \tag{2.10}$$

3. The theory of Ruvinsky & Freidman

We shall now reconcile the analysis of Ruvinsky & Freidman (1985*a*, *b*; 1987) with the above physical argument. These authors formally separate the potential and vortical components of the flow by writing

$$\boldsymbol{u} = \nabla\phi + \boldsymbol{u}', \quad \boldsymbol{u}' = \nabla \wedge \hat{\boldsymbol{\psi}} \tag{3.1}$$

where $\hat{\boldsymbol{\psi}}$ is a vector stream function. They apply to $\boldsymbol{u}'$ a boundary-layer approximation similar to that used in Longuet-Higgins (1953, 1960) and arrive at the following system of coupled equations:

$$\nabla^2\phi = 0, \tag{3.2}$$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + g\eta - \frac{T}{\rho}\kappa(\eta) + 2\nu\frac{\partial^2\phi}{\partial z^2} = 0 \quad \text{on} \quad z = \eta, \tag{3.3}$$

$$\frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial x}\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial z} + v' \quad \text{on} \quad z = \eta, \tag{3.4}$$

$$\frac{\partial\phi}{\partial z} \to 0 \quad \text{as} \quad z \to -\infty, \tag{3.5}$$

together with

$$\frac{\partial v'}{\partial t} = 2\nu \frac{\partial^3 \phi}{\partial x^2 \partial z}\bigg|_{z=\eta}. \tag{3.6}$$

In equation (3.3) $\kappa(\eta)$ denotes the curvature operator: $\partial^2\eta/\partial x^2/(1+\partial\eta/\partial x^2)^{\frac{3}{2}}$.

In the above equations the outstanding coupling term is v' on the right of the kinematic condition (3.4). To remove this we may write

$$\eta = \eta^* + \eta', \quad \eta' = \int v' \,\mathrm{d}t \tag{3.7}$$

and evaluate the boundary conditions on the new surface $\eta = \eta^*$. For simplicity we consider first only the linear terms. Treating η' as of the same order or smaller than η, we see that (3.4) becomes simply

$$\frac{\partial \eta^*}{\partial t} = \frac{\partial \phi}{\partial z}\bigg|_{z=\eta^*} \tag{3.8}$$

without the additional term on the right-hand side. Similarly (3.3) becomes

$$\frac{\partial \phi}{\partial t} + \left(g - \frac{T}{p}\kappa\right)(\eta^* + \eta') + 2\nu \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{3.9}$$

to be satisfied on $z = \eta^*$. But by (3.6)

$$\eta' = \int v' \,\mathrm{d}t = \iint \frac{\partial v'}{\partial t} (\mathrm{d}t)^2 = -\frac{2\nu}{\sigma^2} \frac{\partial^2 \phi}{\partial x^2 \partial z} \tag{3.10}$$

to within a constant. Now operating on both sides of (3.10) by $(g - T/\rho\ \partial^2/\partial x^2)$ and using the dispersion relation (2.6) we see

$$\left(g - \frac{T}{\rho}\kappa\right)\eta' = \frac{\sigma^2}{\kappa}\eta' = -2\nu \frac{\partial^2 \phi}{\partial x^2}. \tag{3.11}$$

So by virtue of Laplace's equation (3.2), equation (3.9) becomes simply

$$\frac{\partial \phi}{\partial t} + \left(g - \frac{T}{\rho}\kappa\right)\eta^* + 4\nu \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{3.12}$$

on $z = \eta^*$. Lastly we note that the term $-4\nu\,\partial^2\phi/\partial z^2$ represents precisely the additional pressure term given by (2.9). For since $\mathrm{i}\sigma = -\partial/\partial t$ and $k = \partial/\partial z$ we have, apart from the constant term,

$$-\delta p = 4\mu k \frac{\partial \eta}{\partial t} = 4\mu k v = 4\mu \frac{\partial^2 \phi}{\partial z^2}. \tag{3.13}$$

We see then that the last term in equation (3.12) represents an additional pressure, half of which comes from the viscous component in the normal stress p_{nn}. The other half comes from the thickening of the vortical boundary layer due to the piling up of mass induced by the tangential stress at the base of the boundary layer.

It is important to recognize that the boundary conditions (3.8) and (3.12) for the potential ϕ are to be evaluated not at the free surface $z = \eta$ but at the modified free surface $z = \eta^*$. After the solution for ϕ is determined, together with η^*, the free surface $z = \eta$ may be recovered by means of (3.7).

4. Discussion

In §3 we simplified the analysis by linearizing the two boundary conditions (3.3) and (3.4). Some linearization is already inherent in any case in the last two terms on the right of these equations. It is not difficult to show that if we retain nonlinear terms on the surface slope ϵ, but not in the ratio kD_0 of the boundary-layer thickness to the wavelength, then (3.8) and (3.12) above are generalized to

$$\frac{\partial \phi}{\partial t}+\frac{1}{2}(\nabla\phi)^2+g\eta^*-\frac{T}{\rho}\kappa(\eta)+4\nu\frac{\partial^2\phi}{\partial n^2}=0 \tag{4.1}$$

and

$$\frac{\partial \eta^*}{\partial t}+\frac{\partial \eta^*}{\partial x}\frac{\partial \phi}{\partial x}=\frac{\partial \phi}{\partial z} \tag{4.2}$$

on $z=\eta^*$, at least to order ϵ^2. At order ϵ^2, we find a contribution to the mass transport and its normal gradient just beneath the vortical boundary layer, which, to this order, may simply be added to the solutions of (4.1) and (4.2). The centrifugal forces associated with the mass transport velocity must however be incorporated into (4.2) at order ϵ^3.

However, the second-order vorticity generated by parasitic capillary waves and released from beneath the boundary layers (Longuet-Higgins 1955, 1960) is much greater than that from the original gravity wave. This vorticity may accumulate very rapidly (in one gravity-wave period) near the crest of the gravity wave and produce a crest vortex. This in turn may significantly affect the dynamics of the parasitic capillaries (see Longuet-Higgins 1991).

We note that all of the above analysis applies to non-breaking and non-turbulent motions in which the kinematic viscosity ν represents the molecular viscosity. It is highly interesting to consider whether an analogous theory might be formulated for breaking waves, in which ν would be replaced by a turbulent eddy coefficient. A full discussion of this question is beyond the scope of the present note, except to remark that generally it will be necessary to include an exchange of mass between the vortical and non-vortical parts of the flow across the lower boundary of the vortical layer. For plunging breakers, a flux of mass and momentum across the upper boundary will also be required. A further requirement is that the vorticity in the surface layer should decay in a time interval of the order of a wave period at most. A residual mean vorticity may however be added in the form of a surface shear current.

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